Math 2050, summary of Week 6

1. Cauchy Sequence

Motivation: How to determine the convergence without discussion on the precise value of limit?

Definition 1.1. *A sequence* $\{x_n\}_{n=1}^{\infty}$ *is said to be Cauchy if* $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ *such that for all* $m, n > N$,

$$|x_m - x_n| < \varepsilon.$$

In other word, instead of controlling the oscillation around the "limit", we control the oscillation between elements!

Expectation: Cauchy sequence is equivalent to convergent sequence!

Recall that a convergent sequence is Necessarily bounded (Bounded Theorem)! And bounded sequence are "almost" convergent by Bolzano-Weierstrass Theorem. We first have:

Lemma 1.1. *A Cauchy sequence is bounded.*

(The proof is essentially the same with Bounded Theorem).

Theorem 1.1 (Cauchy Criterion)**.** *A sequence* $\{x_n\}_{n=1}^{\infty}$ *in* $\mathbb{R}$ *is Cauchy sequence if and only if it is convergent.*

Proof. If $\{x_n\}$ is convergent, then there is $x \in \mathbb{R}$ such that for all $\varepsilon > 0$, there is N such that for all $n > N$,

$$|x_n - x| < \varepsilon/2.$$

Hence, for all $m, n > N$, we have $|x_m - x_n| \leq |x_n - x| + |x_m - x| < \varepsilon$. Therefore it is Cauchy. This proved a easier direction.

For the opposite direction, suppose the sequence is Cauchy. Then it is bounded, hence there is $\{x_{n_k}\}_{k=1}^{\infty}$ such that $x_{n_k} \to x$ for some $x \in \mathbb{R}$ as $k \to +\infty$.Using Cauchy assumption, for all $\varepsilon > 0$, there is N such that for all $m, n > N$,

$$|x_m - x_n| < \varepsilon/2.$$

By replacing m by m_k with $k > N$, we have for all $k, n > N$,

$$|x_n - x_{m_k}| < \varepsilon/2.$$

Since this is true for all $k > N$, we may let $k \to +\infty$ to show

$$|x_n - x| \leq \varepsilon/2$$

for all $n > N$. This completes the proof.

$\square$